https://www.linkedin.com/feed/update/urn:li:activity:6478570132724678657

11. Let $x,y,z > 0$ so that $xy+yz+zx+xyz = 4$. Prove that exists $\triangle ABC$ for which

$$\begin{cases} a = (y+2)(z+2) \\ b = (z+2)(x+2) \\ c = (x+2)(y+2) \end{cases}$$

and show that $1+x+y+z \le xyz + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.

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1. $a+b-c = (y+2)(z+2)+(z+2)(x+2)-(x+2)(y+2) =$
$4z-xy+xz+yz+4 = 4z-xy+xz+yz+xy+yz+zx+xyz =$
$z(y+2)(x+2) > 0$ and, cyclic $b+c-a > 0, c+a-b > 0$.

2. Noting that* $\{(x,y,z) \mid x,y,z > 0$ and $xy+yz+zx+xyz = 4\} =$
$\left\{(x,y,z) \mid (x,y,z) = \left(\frac{2u}{v+w}, \frac{2v}{w+u}, \frac{2w}{u+v}\right), u,v,w > 0 \text{ and } u+v+w = 1\right\}$

we obtain $1+x+y+z \le xyz + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \Leftrightarrow$

(**1**) $\quad 1+\sum \frac{2u}{1-u} \le \frac{8uvw}{(1-u)(1-v)(1-w)} + \sum \frac{1-u}{2u}$.

Let $p := uv+vw+wu, q := uvw$. Then $(1-u)(1-v)(1-w) = p-q$,

$\sum \frac{u}{1-u} = \frac{1-2p+3q}{p-q}$, $\sum \frac{1-u}{u} = \frac{p-3q}{q}$ and inequality (**1**) becomes

$$1+\frac{2(1-2p+3q)}{p-q} \le \frac{8q}{p-q} + \frac{p-3q}{2q} \Leftrightarrow \frac{5}{2} \le \frac{p}{2q} - \frac{2(1-2p-q)}{p-q} \Leftrightarrow$$

$$\frac{5}{2} \le \frac{p}{2q} - 2 - \frac{2(1-3p)}{p-q} \Leftrightarrow \frac{9}{2} \le \frac{p}{2q} - \frac{2(1-3p)}{p-q}.$$

Since $3p \le 1$ $(3(uv+vw+wu) \le (u+v+w)^2 = 1)$ we denote $t := \sqrt{1-3p}$.

Then $p = \frac{1-t^2}{3}$, where $t \in [0,1)$. Since $\frac{p}{2q} - \frac{2(1-3p)}{p-q}$ decrease in $q > 0$ because $q \le p/9$ $(3q^{1/3} = u+v+w = 1$ and $3q^{2/3} = uv+vw+wu = p)$ then, using inequality $q \le \frac{(1-t)^2(1+2t)}{27}$ which give us criteria of solvability Vieta's System $u+v+w = 1, uv+vw+wu = p, uvw = q$ in positive real u,v,w , we obtain

$$\frac{p}{2q} - \frac{2(1-3p)}{p-q} - \frac{9}{2} \ge \frac{\frac{1-t^2}{3}}{2\cdot\frac{(1-t)^2(1+2t)}{27}} - \frac{2\left(1-3\cdot\frac{1-t^2}{3}\right)}{\frac{1-t^2}{3} - \frac{(1-t)^2(1+2t)}{27}} - \frac{9}{2} =$$

$$\frac{9(1+t)}{2(1-t)(1+2t)} - \frac{27t^2}{(1-t)(t+2)^2} - \frac{9}{2} = \frac{9t^2(1-t)}{(2t+1)(t+2)^2} \ge 0.$$

Remark 1.

$xy+yz+zx+xyz = 4 \Leftrightarrow 8+4(x+y+z)+2(xy+yz+zx)+xyz =$
$12+4(x+y+z)+(xy+yz+zx) \Leftrightarrow (x+2)(y+2)(z+2) =$
$(x+2)(y+2)+(y+2)(z+2)+(z+2)(x+2) \Leftrightarrow$

$$\frac{1}{x+2} + \frac{1}{y+2} + \frac{1}{z+2} = 1.$$

Remark 2.

Using upper bounds for q which give us inequality $q \le \frac{p^2}{3}$ and even more sharper inequality $q \le \frac{p^2}{4-3p}$ don't give us desirable result. That is reason why was used the best upper bound $\frac{(1-t)^2(1+2t)}{27}$ for q.

* Easy to see that system of equations $x = \frac{2u}{v+w}, y = \frac{2v}{w+u}, z = \frac{2w}{u+v}$ is solvable in real u, v, w iff $\frac{1}{x+2} + \frac{1}{y+2} + \frac{1}{z+2} = 1$ which is equivalent to $xy + yz + zx + xyz = 4$.